NUMERICAL ANALYSIS OF PLANE POISEUILLE FLOW STABILITY

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 10, No. 4, pp. 115–117, 1969

Study of the stability of plane parallel flows of a viscous incompressible fluid with the aid of the Orr-Sommerfeld equation has found increasing application recently, both to construct the neutral curves and find the critical Reynolds numbers [1] and also for the first attempts to predict theoretically the turbulent mean velocity profiles [2].

The problem reduces mathematically to finding the eigenvalues for the equation

$$\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi = i\alpha R\,[(u - c)(\varphi'' - \alpha^2\varphi) - u''\varphi] \tag{1}$$

with homogeneous boundary conditions for the function φ. Here $u = u(y)$ is the velocity profile of the flow being analyzed for stability; $\varphi = \varphi(y)$ is the complex amplitude of the disturbed motion stream function, having the form $\varphi(y)\exp[i\alpha(x - ct)]$; α is the wavenumber; R is the Reynolds number; c is the unknown eigenvalue. For Im $c > 0$ there is exponential growth of the disturbances (instability); for Im $c < 0$ there is decay.

To date all the numerical methods have made it possible to calculate the eigenvalues of (1) only for comparatively small values of αR—no more than 10^4–10^5.

In [3] a technique for calculating the eigenvalues was proposed which makes it possible to practically remove the limitations on the magnitude of αR and obtain the eigenvalues with a specified precision.

The objective of the present study is to: 1) show the effectiveness of the method of [3] using the example of the study of Poiseuille flow stability in a plane channel, where there is extensive possibility for comparison with the results of other authors; 2) compare the results of the numerical and asymptotic methods over a wide range of values of αR; 3) fill in the gap in the study of plane Poiseuille flow stability—find the dependence of the eigenvalue on the wavenumber α. This last analysis, which is of independent interest as well, may be used to study the nonlinear stability of Poiseuille flow.

The algorithm of [3], somewhat modified, was used to calculate the eigenvalues. The integration of the system of equations was made using the Runge-Kutta method with automatic selection of the step and fixed relative precision. The results relating to the neutral curve were obtained to five significant places, and the other results have at least three significant places.

All the results hereafter are represented in dimensionless parameters, based on the average (discharge) velocity, channel half-width, and molecular viscosity. The eigenvalue is written in the form $c = X + iY$.

Figure 1 shows the "nose" of the neutral curve, computed by the authors (curve 4) for comparison with other results obtained by both asymptotic (curve 1 [4], 2 [1], 3 [5]) and numerical methods (crosses are data of [6], squares are data of [7]).

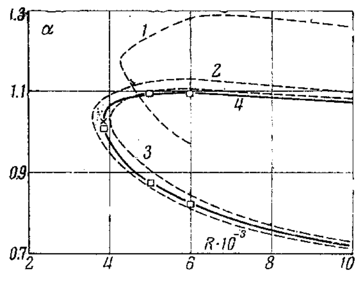

Fig. 1

We see in Fig. 1 that the numerical calculations yield somewhat different results in comparison with the calculations using different asymptotic approximations.

On the other hand, all the numerical results found in the literature agree with the results of the present authors to within the graph precision. For example, for the critical point of the neutral curve

	R_*	α_*	X	Y
[7]	3848.08	1.02071	0.39603	0.00000
Authors	3848.15	1.02041	0.39598	$-0.6\cdot 10^{-8}$

Figure 2 shows the neutral curve (curve 2) calculated by the authors up to $R = 2.5 \cdot 10^8$. The purpose of these calculations was primarily to demonstrate the effectiveness of the method. The computations were made on a BÉSM-6 computer. The neutral points were found by the secant method. Three to five eigenvalues were calculated to find a single neutral point. The computing time for a single eigenvalue in the nose region did not exceed a second.

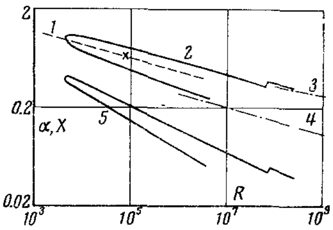

Fig. 2

The numerical calculations agreed exactly with the results of asymptotic theory. The neutral curve in a broad range of Reynolds numbers, obtained by the asymptotic methods, is presented in the survey paper [8]. The dash-dot lines in Fig. 2 show the Lin asymptotes for the upper and lower branches of the neutral curve from the data of [8]: branch 3 – $\alpha = (400/R)^{1/11}$, branch 4 – $\alpha = (142/R)^{1/7}$. Both the asymptotic and the numerical results indicate a "step" in the upper branch at $R = 0.7086 \cdot 10^8$. There is a similar step in the curve for the phase velocity X (curve 5). This is explained in the asymptotic theory by the loop in the Tietjens function [8]. The dashed line in Fig. 2 (curve 1) shows the relation $\alpha(R)$ along the ridge $\max \alpha Y(R)$, which can be approximated for $R > 100$ by the relation $\alpha = (10/3)R^{-1/7}$ to within a percent. The ridge has an apex (shown by the cross in Fig. 2), with the parameters $R = 74 \cdot 10^3$, $\alpha = 0.678$, $X = 0.1897$, $Y = 0.01577$, after which Y decreases approximately according to the law $Y = 1/\pi(\alpha R)^{-1/4}$.

We note that this picture differs considerably from [1], where contours are presented for $Y > 0$ obtained on the basis of the asymptotic theory. With reduction of R along curve 1 in Fig. 2, α approaches 2.81, which corresponds to its value for a stationary fluid.

We have been forced to avoid a tabular representation of the results because of the large volume of data and poor visibility of the tabular form. The following are only the values for the end points of the neutral curve:

	R	α	X	Y
Upper branch	$0.250\cdot 10^9$	0.310	0.0371	$-0.227\cdot 10^{-6}$
Lower branch	$0.290\cdot 10^7$	0.254	0.0538	$-0.172\cdot 10^{-4}$

Lin's asymptotic theory yields values which do not differ graphically from our numerical results for $R \geq 10^5$.

It is of interest to find the dependence of the eigenvalue on the wave number α. To study this dependence over the entire range of wave numbers from zero to infinity, we must combine the numerical calculations with the asymptotic description for very small and very large values of α.

The following asymptotic relations (Fig. 3) are valid for an arbitrary profile $u(y) \in C_2$ for the first eigenvalue:

$$Y = -\pi^2/\alpha R \quad \text{(small } \alpha\text{, curve 6)} \qquad (2)$$

$$Y = -\alpha/R \quad \text{(large } \alpha\text{, curves 7)} \qquad (3)$$

Here the limiting values of the phase velocity X will depend on the form of the profile; for example, for the Poiseuille parabola $X = 0.62$.

In Fig. 3 the solid curves are the numerical results for the dependence of X and Y on α for $R = 10^2$, 10^3, and 10^4 (curves 1, 2, and 3, respectively).

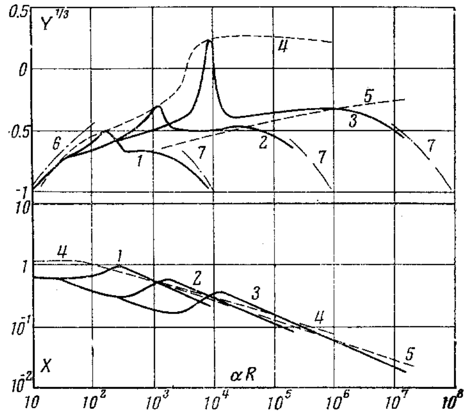

Fig. 3

In the general case, for fixed R, $Y(\alpha)$ has two local maxima. One of them is reached for α of order unity and, beginning with R = 3848, rises above the zero level, forms the neutral curve, and corresponds to instability (curve 4). Another local maximum exists for $R \geq 164$, is located considerably lower, and is reached for α on the order of tens and hundreds (curve 5). The relation X = 1.13, $\alpha RY = -12.96$ holds for curve 4 for small α. For large α the following relations hold quite well along curves 4 and 5:

$$c \approx 7.5\,(\alpha R)^{-1/3} + i\pi^{-1}\,(\alpha R)^{-1/4} \tag{4}$$

$$c \approx (\alpha R)^{-1/3}\,(2\pi - i\,3.86) \tag{5}$$

We note that as $Y(\alpha)$ approaches the asymptotic relation (3) the phase velocities are described well by the relation

$$X \approx \pi^{3/2} R^{-\lg^2} \alpha^{-2/3} \tag{6}$$

(Formulas (4)–(6) were obtained empirically on the basis of the numerical experiments.)

Thus, in the R range studied the eigenvalues (1) for the Poiseuille parabola will be defined.

In conclusion the authors wish to thank M. A. Gol'dshtik for his interest in the study and valuable advice.

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31 December 1968

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